A new method for recovering the space distribution of globular-cluster stars, applied to flare stars in the Pleiades

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A new method is described for recovering the space density of the stars in a globular cluster. The method provides estimates of both the unknown space density and the probable error of recovery, which is statistical in nature and depends only on the population of the cluster. In a typical application, the distribution of 441 flare stars in the Pleiades is found to have a peak of 1.6–2.5 pc^{-3} at the cluster center and to fall off with radius smoothly to zero approximately along a Gaussian curve having a scale parameter of 3.5 pc.

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The problem of recovering the space distribution of the stars in a globular cluster from their projected distribution on the celestial sphere is an important one for clarifying the internal structure of clusters and their course of evolution. A large number of papers have been devoted to this old problem in astronomy over the past 70 years, beginning with the work of von Zeipel[1] and Plummer.[2] Among the later research we would merely mention that of Kholopov[3,4] and Mnatsakanyan.[5]

Mathematically, the problem amounts to solving an integral equation of the first kind:

$$J(R)=2\int_R^{R_0}\frac{I(r)\,r\,dr}{\sqrt{r^2-R^2}},\quad 0\leqslant R\leqslant R_0, \tag{1}$$

in which $J(R)$ represents the surface star density projected on the celestial sphere, $I(r)$ specifies the unknown density in space, and R_0 is the radius of the cluster, that is, a distance such that one may take $I(R) = J(R) = 0$ for $R > R_0$. The change of variables

$$x=1-(R/R_0)^2,\ y=1-(r/R_0)^2,$$
$$j(x)=R_0^2J(R_0\sqrt{1-x}),\ i(y)=R_0^3I(R_0\sqrt{1-y}) \tag{2}$$

transforms Eq. (1) to Abel's integral equation

$$j(x)=\int_0^x\frac{i(y)\,dy}{\sqrt{x-y}},\quad 0\leqslant x\leqslant 1 \tag{3}$$

for the functions $j(x)$ and $i(y)$.

Now before applying some particular method for solving Abel's integral equation numerically, one must estimate the surface density distribution $J(R)$ from the observations. The authors mentioned above all use the histogram method for this purpose: the interval $[0, R_0]$ is divided into a set of intervals each of length ΔR, and a count is then made of the number of stars whose R values lie in each interval. However, as Smirnov[6] demonstrated in 1950, even at best, when the interval $[0, R_0]$ is optimally subdivided into intervals ΔR, the histogram method achieves an accuracy only of about $N^{-1/3}$, where N is the number of stars used to estimate the density. The optimum number of elements in such a histogram is about $N^{1/3}$.

Since the accuracy of recovering the space density $I(r)$ depends chiefly on how accurately the surface density $J(R)$ is determined from the observations, we shall apply not the histogram method but a more accurate procedure suggested by Chentsov,[7] whereby the probability density is estimated from the observations by expanding the unknown density with respect to some orthogonal basis. The statistical precision of this method is approximately $MN^{-1/2}$, where M denotes the number of basis functions used in the expansion. Boyd and Steele[8] have recently shown that no method of estimating the density exists whose accuracy with respect to N is better than $N^{-1/2}$.

As basis functions we shall employ the same functions that we introduced in a previous paper[9] devoted to a numerical solution of Abel's integral equation for the case in which the surface density $J(R)$ is known directly from the observations, along with an experimental uncertainty $\delta J(R)$.

A fundamental feature of our proposed method is that the expansion involves only a finite number of basis functions. Indeed, as indicated above, the statistical accuracy of the method of expansions deteriorates as M increases; but at the same time the transmission of the finer details of the functions $J(R)$ and $I(r)$ improves. The optimum number of terms in the expansion can be determined by minimizing the combined rms error of recovery. If the number of terms in the expansion rises above the optimum, the recovery error will increase. It is the recognition of this fact and the proper choice of the number of terms in the expansion that constitute the difference between our method and the formal method of Picard orthogonal expansions for solving integral equations of the first kind (see Courant and Hilbert[10]).

The final expressions for the space density $I(r)$ and the error $\delta I(r)$ of recovering it take the form

$$I(r)=\frac{1}{2\pi R_0^3}\sum_{k=0}^{M}\frac{\hat{p}_k}{\lambda_k}x^k, \tag{4}$$

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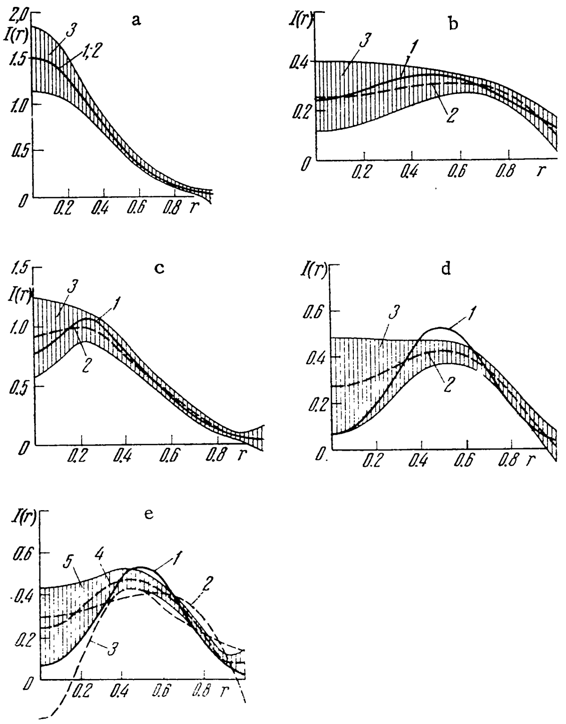

FIG. 1. Tests of the recovery method. a–d) Sample of 441 points: 1) original space density profile; 2) curve fitted to the space density; 3) estimated recovery error. e) samples of differing size: 1) original space density; 2) fit from 20 points; 3) fit from 100 points; 4) fit from 1200 points; 5) estimated recovery error for the 1200-point sample.

where

$$\lambda_0 = 1,\ \lambda_k = \lambda_{k-1}/\ (1 + 1/2k),\quad k = 1, 2, \ldots,\ x = 1 - (r/R_0)^2 \tag{5}$$

and

$$\delta I(r) = \frac{1}{\pi R_0^3}\Big[\sum_{\alpha,\beta=0}^{M} (\hat{D}p)_{\alpha\beta} x^\alpha x^\beta/(\lambda_\alpha\lambda_\beta)\Big]^{1/2}. \tag{6}$$

We have here taken the recovery error to correspond to the 95% confidence interval for the solution I(r). The coefficients $\hat{p}_k$ and the covariance matrix (the error matrix) $\hat{D}_p$ of those coefficients are to be determined observationally.

Figure 1 illustrates the results we have obtained from a Monte Carlo test of the method. For this purpose a random three-dimensional field of points having different distribution laws was generated – distributions both with a central minimum and with a central maximum. The three-dimensional density was then recovered from the projections of these points on a plane. As Fig. 1 shows, the method is indeed capable of restoring the space density for various distribution laws. In most cases the statistical uncertainty of the recovery corresponds to the actual difference between the original density and the density recovered by the method at hand.

For the cases illustrated in Fig. 1, the optimum number of terms in the expansion is no more than seven. Diagrams a–d in this figure have been obtained by using 441 random points, while diagram e shows the results for 20, 100, and 1200 random points. Although the statistical recovery error for 20 points is objectionably large (it is not shown on this graph), comparison of the initial density with the restored density indicates that even with as few as 20 points the method yields acceptable results. We may evidently regard the method as usable in practice beginning with about 50 stars.

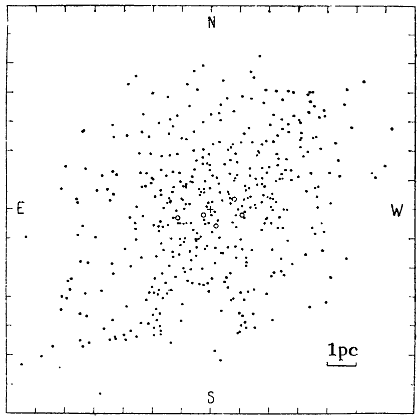

FIG. 2. A chart of the flare stars in the Pleiades. Circles, the five brightest constant stars. The origin ($\alpha = 3^h41^m02^s$, $\delta = +23°53'.3$, 1950.0) is marked by a cross.

Let us apply the method we have described to recover the space density of flare stars in the Pleiades. Figure 2 charts 441 Pleiades flare stars, projected onto a plane tangent to a celestial sphere 126 pc in radius. A scale of 1 pc is given in the lower right corner. The positions of the first 221 flare stars are those given by Ambartsumyan et al.,[11] while the positions of the next 220 flare stars have been taken from supplements to the General Catalog of Variable Stars.[12] Later papers in the series on Pleiades flare stars[13,14] give data up to No. 485 (in the unified numbering adopted by the Byurakan astronomers), but stars Nos. 466–485 have not been included in the analysis. The origin of coordinates in Figs. 2 and 3 is taken to be the point at the center of gravity of the system of points in the plane of projection. Its celestial coordinates are: $\alpha = 3^h41^m02^s$, $\delta = +23°53'.3$ (1950.0).

The cluster center, which we define as the point of maximum surface density, can be determined from Parzen's estimate[15]

$$J(x, y) = \frac{1}{441}\sum_{i=1}^{441} K(x - x_i, y - y_i), \tag{7}$$

in which the "smearing function K(x, y) is taken in the form

$$K(x, y) = \frac{1}{2\pi\sigma^2}\exp[-(x^2 + y^2)/2\sigma^2], \tag{8}$$

and the (x_i, y_i) are the coordinates of the points. Figures 3a and 3b show the contours of equal density of the estimates J(x, y) for two values of the parameter: $\sigma = 0.5$ and 2 pc. The optimum value of the parameter σ evidently lies between these limits. The contour diagrams of Fig. 3 clearly demonstrate that the point of maximum density is very close to the center of gravity of the sys-

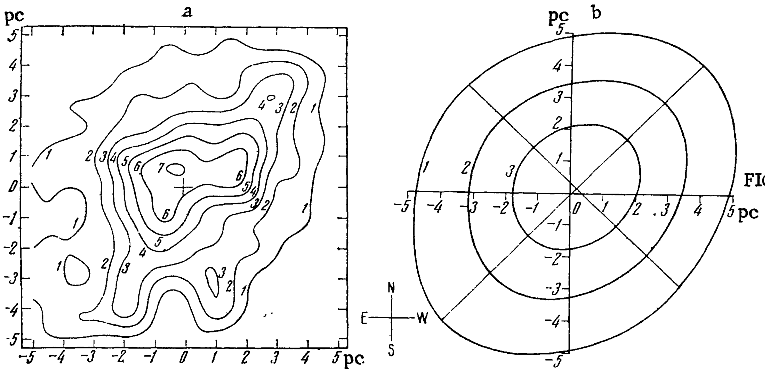

FIG. 3. Contours of equal surface density for: a) σ = 0.5 pc; b) σ = 2pc.

tem of points considered, and that the symmetry of the cluster in the plane of projection is more likely elliptical than circular.

We also arrive at this last result if we count the number of stars in 10 successive 36° sectors relative to the point $\alpha = 3^h40^m47^s$, $\delta = +23°58'.7$ (1950.0), the density maximum in Fig. 3b. Statistically, the angular distribution (Fig. 4) differs significantly from a uniform distribution (which would correspond to a circularly symmetric cluster) but does not differ significantly from a sinusoidal distribution (an elliptically symmetric cluster). According to Kholopov,[4] in the case of elliptical symmetry the space density can be recovered only to within a scale transformation along the r and I(r) axes.

Figure 5 shows the space density profile I(r) for the flare stars in the Pleiades, as deduced from the sample mapped in Fig. 2. To within an accuracy of ≈ 0.05 pc^{-3} (which is better than the recovery error), this curve may be approximated by the expression $I(r) \approx 2 \exp[-(r/r_0)^2]$, in which the characteristic scale r_0 = 3.5 pc. Curiously, this radius r_0 corresponds to the inflection point evident in Fig. 5 and it separates the core and peripheral zones of the cluster. In view of the comparatively small eccentricity of the apparent symmetry ellipse in Fig. 3, we may conjecture that the spatial structure of the flare-star population in the Pleiades conforms nearly to a triaxial ellipsoid whose principal axes are almost equal, with a space density distribution similar to the profile illustrated in Fig. 5.

Our result here is consistent with that of Kholopov,[16] which was obtained without an estimate of the recovery error; but it does not bear out the findings of Mirzoyan and Mnatsakanyan,[17] according to whom the space density at the cluster center is nearly zero.

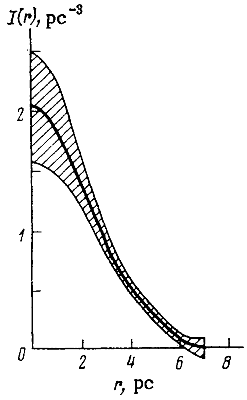

FIG. 5. Heavy curve, space density profiles for 441 flare stars in the Pleiades; shaded band, estimated recovery error.

A more detailed explanation of our method will be published in the variable-star journal Peremennye Zvezdy.

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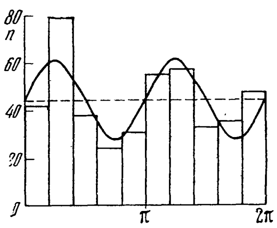

FIG. 4. Angular distribution of the number of flare stars as a function of position angle relative to the point of maximum density. The sine curve represents the best mean-square fit to the histogram.

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