

Heat Transfer in Superfluids: Effect of Gravity

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We discuss the influence of an external field on energy transport in superfluid. He-II is not isothermal in presence of Earth gravity; instead, it supports finite temperature gradient given by a Fourier-like equation. We calculate asymptotic behavior of the effective heat resistance in the vicinity of the λ -transition.

1. INTRODUCTION

Superfluid is known to be (and it was originally referred to as) a *super-heat-conductor*. The heat in superfluid is transferred by the normal flow. This is fundamentally different¹ from the Fourier law thermal conduction in ordinary liquids. Instead, it has a nature of *convection*. In uniform superfluid the flow (and the heat transfer) is dissipationless, *i.e.*, it happens at constant temperature.

Experimentally observed dissipation in bulk uniform superfluid (*i.e.*, when the friction with walls is neglected) is believed to be due to the vortex creation. This scenario² leads to the temperature gradients with complicated nonlinear dependence on the supported heat flux. We show that superfluid on Earth is not isothermal when a heat flow is applied even without vortices.

Earth gravity, like any external field, destroys the system homogeneity: it produces a pressure gradient across the sample. This inhomogeneity is a cause of the normal flow dissipation. In other words, gravity is a friction mechanism between the normal flow and Earth. Due to the friction, the liquid holds a finite temperature gradient whose leading term is proportional to the energy flux. Unlike natural convection in ordinary fluids, this scenario of energy transfer is described by a Fourier-like equation.

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To investigate the effect quantitatively one needs to extend Landau-Khalatnikov superfluid hydrodynamic equations by adding gravity. It is tempting to guess external field contribution to fluid dynamics by inserting appropriate terms to the conservation laws these equations are based upon. The form of dissipative terms however is not self-evident in this approach. To ascertain correct appearance of the equations we use (Sec.2.) standard Landau-Khalatnikov theory in a free-fall frame of reference. Translating all terms to the laboratory frame we unambiguously get sought equations.

In Sec.3. we apply obtained equations to the problem of energy transfer in superfluid. For simplicity we restrict ourselves to one-dimensional vertical setup where influence of walls is neglected and the energy goes either up or down. With the help of linearized hydrodynamic equations we find temperature gradient induced by the heat flux. In linear regime the flux and the gradient are proportional to each other. We conclude the section by an expression for the effective heat resistance of this system.

Effective heat resistance derived in Sec.3. becomes larger in the vicinity of the critical temperature. In Sec.4. we show that it diverges at λ -point and calculate its critical behavior.

2. HYDRODYNAMICS

Ordinary superfluid hydrodynamic equations are modified by external fields. To find out the effect of gravity we use the equivalence principle of general relativity. For the uniform gravity field of Earth $\mathbf{g}$ we define a non-inertial free-fall frame of reference $\tilde{K}$. It has constant acceleration $\mathbf{g}$ with respect to the laboratory frame $K(t, x^i)$. We also demand that at $t = 0$ this frame coincides with K and has zero relative velocity. This ensures that hydrodynamic variables at this moment are equal* in two frames. In the frame $\tilde{K}$ no gravity is present and (in Newtonian limit) the superfluid is governed by conventional equations³

$$\left\{ \begin{array}{l} \frac{\partial \tilde{\rho}}{\partial t} = -\frac{\partial \tilde{j}^i}{\partial x^i} \\ \frac{\partial \tilde{j}^k}{\partial t} = -\frac{\partial \tilde{\Pi}^{ik}}{\partial x^i} \\ \frac{\partial \tilde{v}_s^k}{\partial t} = -\frac{\partial \tilde{\Phi}}{\partial x^k} \\ \frac{\partial \tilde{E}}{\partial t} = -\frac{\partial \tilde{Q}^i}{\partial x^i}, \end{array} \right. \quad (1)$$

*They are functions of ρ , $\mathbf{j}$, $\mathbf{v}_s$, and E but not of their time derivatives.

where tilde attributes quantities to $\tilde{K}$, ρ is the fluid density, $\mathbf{j}$ is the mass flux, $\mathbf{v}_s$ is the superfluid flow velocity, and E is the energy density. The fluxes in (1) are given by

$$\tilde{\Pi}^{ik} = v_s^i j^k + v_n^k j^i - \rho v_n^k v_s^i + p \delta_{ik} + \tilde{\pi}^{ik}, \quad (2)$$

$$\tilde{\pi}^{ik} = -\eta \left(\frac{\partial v_n^i}{\partial x^k} + \frac{\partial v_n^k}{\partial x^i} - \frac{2}{3} \frac{\partial v_n^l}{\partial x^l} \delta^{ik} \right) - \delta^{ik} \left(\zeta_1 \frac{\partial(j^i - \rho v_n^i)}{\partial x^i} + \zeta_2 \frac{\partial v_n^l}{\partial x^l} \right), \quad (3)$$

$$\tilde{\Phi} = \mu + v_s^2/2 + \tilde{\phi}, \quad (4)$$

$$\tilde{\phi} = -\zeta_3 \frac{\partial(j^i - \rho v_n^i)}{\partial x^i} - \zeta_1 \frac{\partial v_n^i}{\partial x^i}, \quad (5)$$

and

$$\tilde{Q}^i = \left(\mu + \frac{v_s^2}{2} \right) j^i + ST v_n^i + v_n^i v_n^k \left(j^k - \rho v_s^k \right) + (j^i - \rho v_n^i) \tilde{\phi} + v_n^k \tilde{\pi}^{ik} - \kappa \frac{\partial T}{\partial x^i}. \quad (6)$$

Here $\mathbf{v}_n$ is the velocity of the normal flow, p is the pressure, μ is the chemical potential, S is the entropy density, T is the temperature, and η , ζ_1 , ζ_2 , ζ_3 , κ are kinetic coefficients. As usual, we took not all the dissipative terms that are in principle possible, but only the largest. In general the dissipation in isotropic superfluid is described by 13 kinetic coefficients rather than by 5 (A. Clark 1963, see discussion in Ref.1, p.525).

To rewrite the equations in laboratory frame of reference one must translate left-hand side terms in (1) to K . This is done by transformations

$$\begin{aligned} \tilde{\rho} &= \rho, & \tilde{\mathbf{j}} &= \mathbf{j} - \mathbf{g}t\rho, \\ \tilde{\mathbf{v}}_s &= \mathbf{v}_s - \mathbf{g}t, & \tilde{E} &= E - \mathbf{j}\mathbf{g}t + \rho g^2 t^2/2. \end{aligned}$$

At $t = 0$ one finally gets

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial j^i}{\partial x^i} = 0 \\ \frac{\partial j^k}{\partial t} + \frac{\partial \tilde{\Pi}^{ik}}{\partial x^i} = \rho g^k \\ \frac{\partial v_s^k}{\partial t} + \frac{\partial \tilde{\Phi}}{\partial x^k} = g^k \\ \frac{\partial E}{\partial t} + \frac{\partial \tilde{Q}^i}{\partial x^i} = j^k g^k. \end{cases} \quad (7)$$

3. EFFECTIVE HEAT RESISTANCE

For the steady-state flow all time derivatives in (7) vanish. Consider one-dimensional system where all velocities and fluxes are vertical. Let z -axis run down along $\mathbf{g}$. In typical heat transfer experiment⁴ the energy is carried by the superfluid counterflow, where normal and superfluid velocities are directed oppositely to each other to keep zero net mass flux. The condition $j = 0$ is satisfied along the height of the cell. Under these circumstances the first equation in (7) is trivially satisfied and the other three have the form

$$\begin{cases} \Pi' = \rho g \\ \phi' = g \\ Q' = 0, \end{cases} \quad (8)$$

where primes denote the $\partial/\partial z$ derivative.

Within linear (on velocity and temperature gradient) approximation the fluxes in (8) are given by

$$\Pi = p + \zeta_1(\rho v_n)' - \zeta_2 v_n' - \frac{4}{3}\eta v_n', \quad (9)$$

$$\phi = \mu + \zeta_3(\rho v_n)' - \zeta_1 v_n', \quad (10)$$

$$Q = TS v_n - \kappa T'. \quad (11)$$

Keeping lowest nontrivial gradient terms (thus assuming the effect of $\mathbf{g}$ to be small) and excluding μ' by means of thermodynamic identity $\rho d\mu = dp - S dT$, we obtain from (8), (9) and (10)

$$A' v_n' + B v_n + S T' = 0,$$

where we introduced

$$A = 2\rho\zeta_1 - \frac{4}{3}\eta - \zeta_2 - \rho^2\zeta_3,$$

$$B = (\zeta_1' - \rho\zeta_3') \rho' + (\zeta_1 - \rho\zeta_3) \rho''.$$

Finally employing (8) and (11) we obtain a Fourier-like equation

$$T' = -\varrho Q, \quad (12)$$

where the effective heat resistance ϱ is determined by

$$\varrho = \frac{1}{TS^2} \left(B - \frac{S'}{S} A' \right). \quad (13)$$

We see that in linear approximation the temperature gradient is proportional to the heat flux. The effective heat resistance is of the second order with respect to system inhomogeneity, *i.e.*, $\varrho \propto g^2$. If higher terms are taken into account, the same heat flow applied from above and from below will lead to different temperature gradients.

4. CRITICAL BEHAVIOR

The effect discussed in previous section is more pronounced in the vicinity of the λ -point. To find critical behavior of the effective heat resistance we need to estimate kinetic coefficients entering (13).

This can be done using the ideas of dynamic scaling⁵. The specific heat index for helium is very small⁶, we further let it be equal to zero. The superfluid density ρ_s follows Josephson scaling relation⁷ $\rho_s \propto r_c^{-1} \propto t^{2/3}$, where r_c is the correlation length and t is the reduced temperature $t = T_\lambda - T^\dagger$.

Second sound damping³ at the boundary of the critical region is inversely proportional to the correlation length

$$r_c^{-1} \propto \text{Im} \frac{\omega}{u_2} = \frac{\omega^2}{2\rho u_2^3} \left(\frac{\rho_s}{\rho_n} \left(\frac{4}{3}\eta + \zeta_2 + \rho^2 \zeta_3 - 2\rho \zeta_1 \right) + \frac{\kappa \rho}{T} \frac{\partial T}{\partial S} \right). \quad (14)$$

Here $\rho_n = \rho - \rho_s$, ω and u_2 are the complex frequency and the velocity of the second sound respectively. The latter is determined by the expression

$$u_2 = \frac{S}{\rho^{1/2}} \sqrt{\frac{\partial T}{\partial S} \frac{\rho_s}{\rho_n}} \propto t^{1/3}.$$

Further assuming that at the critical region boundary all terms in (14) make contribution of the same order to the second sound damping we get for the kinetic coefficients

$$\eta \propto \zeta_1 \propto \zeta_2 \propto \zeta_3 \propto r_c u_2 \rho_s^{-1} \propto t^{-1}. \quad (15)$$

To find the main term of the space derivatives in (13) we note that

$$\frac{\partial}{\partial z} \propto \rho g \left(\frac{\partial}{\partial p} \right)_T \propto -\frac{\rho g}{T} \frac{dT_\lambda}{dp} \left(\frac{\partial}{\partial t} \right)_p.$$

[†]The same letter t is used to represent the time and the reduced temperature. This should not lead to confusion however: the meaning is clear from the context. Another popular choice ϵ seems to be prone to clash with energy due to coincidence of dimensionality.

Here the derivative dT_λ/dp is taken along the phase transition curve. Substituting this in (13) and using (15) we get[‡]

$$\varrho \propto \frac{g^2}{t^2}. \quad (16)$$

Having shown that finite temperature gradient in superfluid may exist, it is instructive to find the maximum gradient possible. Obtained results on effective heat resistance (being derived within linear approximation) correspond to small temperature gradient and velocities. They are therefore not applicable to the critical velocities region. We will try, however, to use them as an estimate beyond the limits. It is shown in Ref. 9 that the critical energy flux Q_c in the vicinity of the λ -transition behaves roughly as $Q_c \propto t^{4/3}$. Using (16) we see that the maximum temperature gradient T'_c increases when the temperature approaches T_λ as $T'_c \propto t^{-2/3}$.

ACKNOWLEDGMENTS

The author is indebted to A.F.Andreev for attention to this work and to R.V.Duncan and D.A.Sergatskov for fruitful discussions. This work was supported in parts by INTAS grant 01-686, CRDF grant RP1-2411-MO-02, RFBR grant 03-02-16401, and RF president program.

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[‡]It should be emphasized that our accuracy does not allow to resolve between T_λ and $T_c(Q)$ introduced by some investigators⁸. In linear approximation these two temperature are identical.