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Nonlinear Green's Functions in Smectics

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A solution of the problem of the strain of smectics subjected to a localized force has been proposed. An increase in the force is accompanied by a change from a linear increase in the strain amplitude to a square-root increase.

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The equation of the elastic equilibrium of smectics,

$$\begin{aligned}\lambda^2\Delta_\perp^2 u - \partial_z^2 u + \partial_z(\partial_\alpha u)^2 + (\partial_z u)(\Delta_\perp u) \\ -\partial_\alpha[(\partial_\beta u)^2\partial_\alpha u]/2 = F(\mathbf{r})/A\end{aligned} \quad (1)$$

is obtained when varying the strain energy given by Eq. (44.13) in [1]. Here, u is the shift of the layers along the smectic axis z, A is the elastic modulus, λ is the microscopic length parameter, ∂_α is the gradient vector in the xy layer plane, $\Delta_\perp = \partial_\alpha^2$, and $F(\mathbf{r})$ is the external force density.

For the case of the localized force distribution with the integral F, the linear approximation far from the force application point provides [2]

$$u = h\ln\frac{d}{\rho} - h\int_{\rho^2/\lambda|z|}^{\infty} e^{-\xi/4}\frac{d\xi}{2\xi}. \quad (2)$$

Here, $\rho = \sqrt{x^2+y^2}$ and $h = \lambda F/F_c$, where $F_c = 4\pi A\lambda^2$. Asymptotic expression (2) is valid at distances $|z| \ll L$ and $\rho \ll d \sim \sqrt{\lambda L}$, where L is the distance from the force application point to the smectic boundary along the z axis. Boundary conditions become significant beyond this region.

However, as shown in our previous works [2, 3], the long-range asymptotic expressions in a number of problems in small-angle approximation (1) are determined by nonlinear effects. This phenomenon for the edge dislocation [3] was observed by Ishikawa and Lavrentovich [4]. As will be seen, solution (2) is valid only in the limit $F \ll F_c$ (usually, $F_c \sim 10^{-5}$–10^{-4} dyne).

Let us show that the nonlinear problem of a single force has a solution in the form $u = h\ln(d/\rho) + f(v)$, where $v = \rho^2/z$ (the F direction is taken as the positive direction of the z axis; in this case, F, $h > 0$). Then, Eq. (1) is reduced to the form

$$\left(16\lambda^2 v\psi'' - v\psi + 4h\psi - 6\psi^2 + \frac{(h-2\psi)^3}{v}\right)' = 0, \quad (3)$$

where $\psi = vf'$. Thus,

$$16\lambda^2 v\psi'' - v\psi + 4h\psi - 6\psi^2 + \frac{(h-2\psi)^3}{v} = C, \quad (4)$$

where C is the constant. We are interested in the solution of Eq. (4) without singularity at $v \longrightarrow 0$. It is easy to verify that such a solution exists only under the conditions $\psi \longrightarrow h/2$ and $C = h^2/2$. In the new notation, $\varphi = \psi/h$ and $x = v/h$, we obtain

$$\frac{16\lambda^2}{h^2}x\varphi'' = (4\varphi - 2 + x)\left(\frac{(1-2\varphi)^2}{2x} + \varphi\right). \quad (5)$$

The second factor on the right-hand side of this equation determines the stress field σ_{zz}, which should satisfy the integral force balance relation $F_+ + F_- = F$, where

$$F_\pm = \pm\int\sigma_{zz}dS = \pm A\int\left(\frac{(\partial_\alpha u)^2}{2} - \partial_z u\right)dS. \quad (6)$$

The integrals F_+ and F_- are calculated at certain values $z > 0$ and $z < 0$, respectively. In terms of φ, we have

$$F = \pi A h^2\int_{-\infty}^{+\infty}\left(\frac{(1-2\varphi)^2}{2x} + \varphi\right)dx. \quad (7)$$

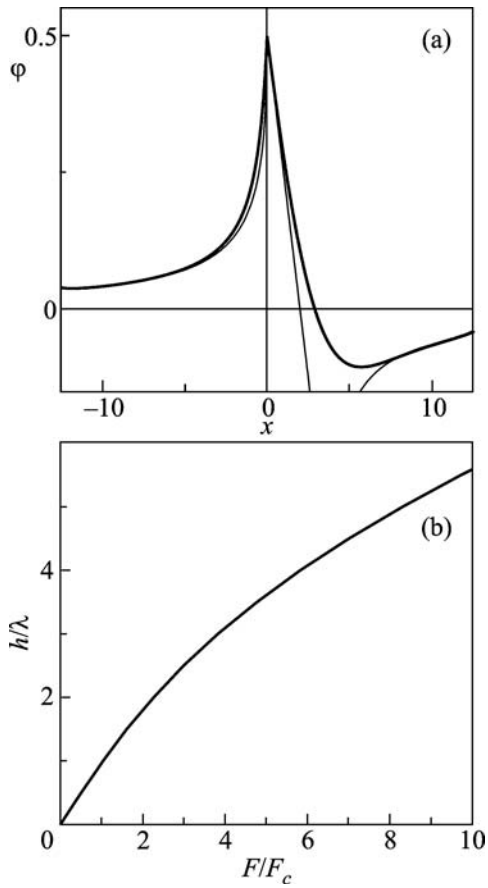

Fig. 1. (a) (The thick line) The function $\varphi(x)$ at $h = 4\lambda$ and (the thin line) the analytical limit at $h \gg \lambda$; the function is symmetric, $\varphi = 0.5\exp(-h|x|/4\lambda)$ in linear approximation (2). (b) The function $h(F)$.

Therefore, the solution at $|x| \gg 1$ should tend to one of the zeros of the second factor on the right-hand side of Eq. (5). Since the inclination of the smectic layers should vanish far from the force application point, $\varphi \longrightarrow -1/2x$ at $|x| \gg 1$.

In the limit $h \gg \lambda$, the term with the second derivative in Eq. (5) is significant only in the regions of sharp change in the function φ. Beyond these regions, the solution differs only slightly from one of three functions $\varphi_0 = (2 - x)/4$ and $\varphi_\pm = \{2 - x \pm \sqrt{x(x-4)}\}/4$ with which the right-hand side of Eq. (5) vanishes.

The behaviors of the displacement field for opposite signs of x are significantly different. At $x > 4$, $\varphi \approx \varphi_+$, and $\varphi \approx \varphi_0$ in the range $0 < x < 4$. Equation (5) can be simplified near the point $x = 4$: $\eta'' = \eta(\eta^2 - t)$, where $\eta = 2^{-1/6}(h/\lambda)^{1/3}(\phi - \phi_0)$ and $t = 2^{-8/3}(h/\lambda)^{2/3}(x - 4)$. Details of the behavior of the function φ are insignificant for the calculation of F_+. The integral F_+ is primarily collected in the region $x < 4$ and is $F_+ = \pi A h^2$.

At $x < 0$, $\varphi = \varphi_- + \chi$, where $\chi \ll \varphi_-$. To calculate F_-, it is necessary to determine the correction χ. At $|x| \gg (\lambda/h)^2$, the left-hand side of Eq. (5) can be treated as a perturbation. Then, $\chi = (4\lambda/h)^2(-x)^{-1/2}(4 - x)^{-5/2}$. In this case, integral F_- is logarithmically divergent at small x values, $F_- = \pi A\lambda^2\{2\ln(2h/\lambda) - 1 + c\}$. The constant c is determined by the solution at $|x| \sim (\lambda/h)^2$, where perturbation theory is inapplicable. In this region, Eq. (5) is simplified to the form $t^2\eta'' = \eta(\eta^2 - t)$, where $\eta = h(\varphi - \varphi_0)/\sqrt{2}\lambda$ and $t = -h^2x/8\lambda^2$. The numerical solution provides $c \approx 1$.

Thus, the stretching effect at $z < 0$ can be neglected in the force balance $F \gg F_c$ as compared to the compression effect $\pi A h^2 \approx F$ at $z > 0$. As a result, we arrive at the law $h = \sqrt{F/\pi A}$.

The displacement field is easily reconstructed from functions φ_0 and $\varphi_\pm$. In the ordinary variables at $z > 0$ and $\rho > 2\sqrt{hz}$, as well as at $z < 0$ and $\rho \gg \lambda\sqrt{|z|/h}$,

$$u = h\ln\frac{2d}{\sqrt{\rho^2 - 4hz} + \rho} + \frac{\rho}{4z}\left(\sqrt{\rho^2 - 4hz} - \rho\right) + \frac{h}{2}.$$

At $z > 0$, $u = h\ln(d/\sqrt{hz}) - \rho^2/4z + h/2$ in the range $0 < \rho < 2\sqrt{hz}$; at $z < 0$, the transition to the law $u = h\ln(d/\sqrt{h|z|}) + h/2$ at $\rho \ll \lambda\sqrt{|z|/h}$ occurs at $\rho \sim \lambda\sqrt{|z|/h}$. The asymptotic expression obtained for the displacement field is valid at $|z| \ll L$ and $\rho \ll d \sim \sqrt{hL}$.

At $0 < F < 10F_c$, the problem was solved numerically (see figure) by the shooting method from the point $x = 0$, near which $\varphi = 1/2 + \gamma x$. The parameter γ (which depends on the sign of x) was chosen so as to ensure the correct asymptotic behavior $\varphi = -1/2x$ at large $|x|$ values.

Note that the smectic layer profile at $z = 0$ has the form $u = h\ln(d/\rho)$; here, $\sigma_{zz} = 0$ beyond the force application point ($\rho = 0$). For this reason, the results can be expanded on the problem of the force acting on the base surface of the smectic. Indeed, the strain of the surface of a normal liquid subjected to the force F_s is $u = (F_s/2\pi\sigma)\ln(R/\rho)$, where σ is the surface tension (the liquid occupies the half-space $z < 0$). The force F in the smectic is decomposed into the surface and bulk components. The surface component F_s is balanced with the surface tension similar to the case of the normal liquid (note that the surface tension is equal to the surface energy only on the base surface of the smectic). From the condition of the coincidence of two expressions for the surface profile, we obtain $F_s = 2\pi\sigma h$ and $R = d$.

In the linear approximation, the bulk force component is $F_v = 2\pi A\lambda h$ (recall that strains appear only in a half-space). Thus, $2\pi\sigma h + 2\pi A\lambda h = F$ and the surface profile amplitude changes sign with a changing force sign. As the force increases, the behavior of the smectic becomes asymmetric. Under the application of a large force directed inside the smectic (compression, i.e., $F < 0$ and $h < 0$), $2\pi\sigma h - \pi A h^2 = F$. If the force has the opposite direction (stretching, i.e., $F > 0$ and $h > 0$), then $2\pi\sigma h + 2\pi A\lambda^2 \ln(2h/\lambda) = F$.

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