Infinite-dimensional representations of the Einstein group

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The representations of the general linear group $GL(4.R)$ are described. This group corresponds to the space–time transformations discussed in the general theory of relativity. Besides the well-known tensor representations, the group is also characterized by infinite-dimensional representations with integral and half-integral spins. This fact opens up a natural possibility, in principle, of constructing a covariant theory of particle fields. © *1996 American Institute of Physics.* [S0021-3640(96)00217-4]

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The general theory of relativity[1] is based on the analysis of continuous transformations of four-dimensional space–time that engender a group of linear transformations for the differentials of the coordinates

$$dx^i = a^i_k dx'^k, \tag{1}$$

where a^i_k is a real 4×4 matrix:

$$a^i_k = \frac{\partial x^i}{\partial x'^k}. \tag{1a}$$

In mathematics a group of transformations of the form (1) is called the general linear group $GL(4.R)$. In physics it is natural to call it the Einstein group. The group of spatial rotations and the Lorentz group are, evidently, subgroups of the Einstein group.

All finite-dimensional representations of the group $GL(N.R)$ have been constructed by Gel'fand and Tseĭtlin.[2] In our case $GL(4.R)$ these representations are apparently equivalent to tensor representations. The fact that the standard spinor representations (spin 1/2) of the Lorentz group cannot be generalized within the Einstein group became known almost immediately after the appearance of the Dirac equation. This problem was overcome by different methods, but they can hardly be regarded as natural. In the present letter, attention is drawn to the infinite-dimensional representations of the Einstein group. They include representations with half-integer spins.

As is well known (see, for example, Ref. 3), the irreducible representations of the Lorentz group are determined by a pair of numbers (k_0, c), where k_0 is an integral or half-integral non-negative number and c is a complex number. For an appropriate choice of the basis $f_{k,\nu}$ in the space of representations the irreducible representation corresponding to a given pair (k_0, c) is given by the formulas

$$
\begin{aligned}
&L_+ f_{k\nu} = \sqrt{k+\nu+1}\sqrt{k-\nu}\; f_{k,\nu+1}, \\
&L_- f_{k\nu} = \sqrt{k+\nu}\sqrt{k-\nu+1}\; f_{k,\nu-1}, \quad L_3 f_{k\nu} = \nu f_{k\nu}, \\
&F_3 f_{k\nu} = C_{k\nu}\; f_{k-1,\nu} - \nu \frac{i k_0 c}{k(k+1)} f_{k\nu} - C_{k+1,\nu}\; f_{k+1,\nu}, \\
&C_{k\nu} = \frac{\sqrt{k^2-\nu^2}\sqrt{k^2-k_0^2}\sqrt{c^2-k^2}}{k\sqrt{4k^2-1}},
\end{aligned}
\tag{3}
$$

where L_+, L_-, and L_3 are the spatial rotation operators, and F_3 is a Lorentzian rotation operator in the (t,z) plane. The Lorentzian rotations in other planes can be easily reconstructed from the commutation relations between the Lorentzian and spatial rotations. If $c^2 = (k_0+n)^2$ for some positive integer n, then the representation is finite-dimensional and the indices ν and k take on the values $\nu = -k, -k+1, \ldots, k$ and $k = k_0, k_0+1, \ldots, k_0+n-1$. If, however, $c^2 \neq (k_0+n)^2$ for any positive integer n, then the representation is infinite-dimensional and ν and k take on the values $\nu = -k,$ $-k+1, \ldots, k$ and $k = k_0, k_0+1, k_0+2, \ldots$.

It is convenient to choose the unit matrix as one of the ten matrices in the complement of the Lorentz group with respect to the Einstein group — this matrix describes the general change in the scale of space–time. Since this matrix commutes with arbitrary matrices, it is always possible to choose a basis such that all functions of a given representation will be eigenfunctions of a corresponding operator M with the same eigenvalue μ. For example, for a vector representation $\mu = 1$, for a covariant vector $\mu = -1$, and for a tensor representation μ equals the rank of the tensor. In the general case, μ is a complex number.

I note here that if attention is confined to transformations of the Lorentz group and to a scale transformation M, then in complete analogy with the principles of the general theory of relativity it is possible to construct a truncated version of the theory in which the classification of the representations of the Lorentz group is preserved completely and, to a first approximation, the gravitation will coincide with the Newtonian limit, the helicity of gravitons will equal zero, and the theory will not contain any black-hole type singularities of the metric. However, the corrections to the Newtonian approximation are different from the Einstein corrections and as a result, specifically, the numerical coefficients in the refraction of a light beam in a gravitational field and the secular shift of the perihelion of orbits are different. While there is no doubt now that the general theory of relativity predicts the correct magnitudes of these effects, the truncated version of the theory does not agree with nature.

Choosing the nine remaining matrices somehow or other, it is easy to show that it is sufficient to determine one of the corresponding operators, whereupon the action of the remaining eight operators is easily determined with the aid of the commutation relations with the operators of the Lorentz group.

Investigation shows that only the representations of the Lorentz group for which one of the numbers k_0 or c equals zero can be completed up to the representations of the Einstein group. Since formulas (2) are symmetric with respect to an interchange of k_0 and c, we introduce a single letter, s, which can now be any complex number, to denote such representations of the Lorentz group. If s equals an integer or half-integer, then the functions which realize either the $(0,s)$ or $(s,0)$ representations of the Lorentz group participate in the formulas presented below and in all other cases only the $(0, s)$ representations participate.

For an appropriate choice of basis of the irreducible representation, the action of the operator A_3 corresponding to an infinitesimal orthogonal rotation in the (t, z) plane by an angle $\delta\chi$,

$$\delta t = \delta\chi z, \quad \delta z = -\delta\chi t,$$

is determined by the formulas

$$\delta f_{sk\nu} = \delta\chi A_3 f_{sk\nu},$$

$$\begin{aligned} A_3 f_{sk\nu} = {} & D_s \sqrt{k-s}\sqrt{k+s+1}\sqrt{k+s+2}\sqrt{k+s+3} B_{k+1,\nu}\, f_{s+2,k+1,\nu} \\ & + D_s \sqrt{k-s-1}\sqrt{k-s-2}\sqrt{k-s}\sqrt{k+s+1} B_{k,\nu}\, f_{s+2,k-1,\nu} \\ & + D_{s-2} \sqrt{k-s+2}\sqrt{k-s+1}\sqrt{k-s+3}\sqrt{k+s} B_{k+1,\nu}\, f_{s-2,k+1,\nu} \\ & + D_{s-2} \sqrt{k-s+1}\sqrt{k+s-2}\sqrt{k+s-1}\sqrt{k+s} B_{k,\nu}\, f_{s-2,k-1,\nu} \\ & + \alpha \frac{k+1}{s^2-1}\sqrt{(k+1)^2-s^2} B_{k+1,\nu} f_{s,k+1,\nu} + \alpha \frac{k}{s^2-1}\sqrt{k^2-s^2} B_{k,\nu}\, f_{s,k-1,\nu}, \end{aligned} \tag{4}$$

$$B_{k\nu} = \sqrt{\frac{k^2-\nu^2}{4k^2-1}}, \quad D_s = \frac{\sqrt{\alpha^2-(s+1)^2}}{2(s+1)\sqrt{s(s+2)}}.$$

For the $(s, 0)$ representations of the Lorentz group with half-integral s the parameter α can be an arbitrary complex number, and for integral s the parameter α can assume only integer values greater than 1. For representations of the form $(0, s)$ the parameter α can be an arbitrary complex number, and for positive integral s, if $\alpha = s+n$, where $n>1$ is an odd number, the representation is finite-dimensional and, as is easily verified by a direct comparison, it is equivalent to a tensor representation.

The result (4) was obtained by the method presented in Ref. 3 for examples of the rotation and Lorentz groups. The scheme of the solution is as follows. Let the operator A_3 have the matrix form

$$A_3 f_{\alpha k_0 c k \nu} = A_{\alpha k_0 c k \nu}^{\alpha' k_0' c' k' \nu'} f_{\alpha' k_0' c' k' \nu'}. \tag{5}$$

Here, besides the numbers k_0, k, and ν characterizing the functions in the basis of the irreducible representations of the Lorentz group, an additional integral index α which enumerates the functions that realize the equivalent irreducible representations of the Lorentz group has been introduced (the multiplicity of the representations of the Lorentz group in the representations of the Einstein group is not obvious in advance). Note that the multiplicity of the representations of the rotation group in infinite-dimensional (finite-dimensional) representations of the Einstein group increases (decreases) linearly with increasing spin. We now write out the equations for the matrix elements (5) that are equivalent to the commutation relations of the 4×4 matrices of the Einstein group. Equations which are linear and quadratic in A are obtained. From the condition that orthogonal rotations in the (t,z) and (x,y) planes commute it follows that $\nu'=\nu$. The linear equations give the dependence of A on the index ν. From the condition that the linear equations be consistent we find that either k_0 or c must equal zero (we denote by s the remaining nonzero number) and that the selection rules

$$k=k\pm1,\quad s=s\pm2, \tag{6}$$

hold, and we also determine the dependence on the index k. Analysis of the quadratic equations shows that for an appropriate choice of the functions f the matrix A can be diagonalized with respect to the index α, i.e., the multiplicity of the representations of the Lorentz group in the representations of the Einstein group equals one. Then, the dependence of the elements A on the index s can be easily determined from the quadratic equations.

For the sake of mathematical rigor, it should be stated that the present investigation leaves open the question of whether or not a representation of the Einstein group corresponds to each pair of numbers $(\alpha,\ \mu)$. However, this question requires a special analysis by different methods.

We note that the numbers α and μ, which fix the representation, have the meaning of quantum numbers (completely analogous to spin) — definite conservation laws must be associated with them, since these numbers determine the structure of the invariants in the Lagrangian and it is possible that some of the conserved charges characterizing particles reduce to them.

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